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ON THE PROBLEM OF EDGE-BONDED ELASTIC QUARTER-PLANES LOADED AT THE BOUNDARY

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Abstract—The stress field in bonded quarter-planes of different elastic materials due to arbitrary (integrable) normal and shear loading applied at the boundary is given explicitly in terms of the two composite parameters α, β introduced in Dundur's discussion [2] of the author's previous paper [1].

The singularity in the stress field is studied for all physically relevant values of α, β. The singularity is of order $r^{-\lambda}$, $\log r$, or 1 depending on the values of α, β. The solution for uniform normal and shearing tractions is also obtained from a limiting case.

The general solution is also used to derive a simple algebraic function of α only, that represents the fraction of the applied load borne by each quarter-plane.

Finally the loading is specialized to that of a concentrated normal force and the resulting stress components are shown graphically as a function of position along the bonded edge for various values of α, β.

1. INTRODUCTION

In a recent paper [1] the general problem of edge-bonded quarter-planes of dissimilar isotropic elastic materials loaded by arbitrary integrable boundary tractions was considered (see Fig. 1). The Mellin transform was used in a straight-forward manner to obtain the solution; the most difficult task was the algebraically complex one of finding the solution, in explicit functional form, of a system of eight linear algebraic equations with coefficients depending on the four elastic constants. A minor simplification resulted due to the fact that the four elastic constants entered as only three parameters since the two shear moduli appeared as a ratio. The solution was recorded in [1] in the most concise manner, i.e. in terms of the transforms of the two Airy functions appropriate to the two quarter-planes. From these functions the transforms of the stress and displacement fields follow by simple differentiation and the components of the corresponding physical quantities then result from the appropriate Mellin inversion integrals.

This solution was used to investigate the nature of the singularity in the stress field, which occurs at the intersection of the bonded edge with the loaded boundary (at $r = 0$). It was found that a singularity of the order $r^{-\lambda}$, $\lambda > 0$ (non-oscillatory) where λ depends on the elastic constants, occurs for certain combinations of materials, but that no such singularity would occur for other combinations. Graphical results were given which showed† the dependence of λ on the ratio of the shear moduli μ'/μ'' for particular chosen values of the two Poisson's ratios ν', ν''. These results were directly related to the roots of a certain transcendental equation in the complex transform plane, which had to be solved numerically. The asymptotic behavior of the stress field as $r \to 0$, when the singularity is of order less than $r^{-\lambda}$ for any $\lambda > 0$, was not included since this behavior depends on the detailed form of the

† In [1] the power λ was denoted by $\sigma_1 + 2$.

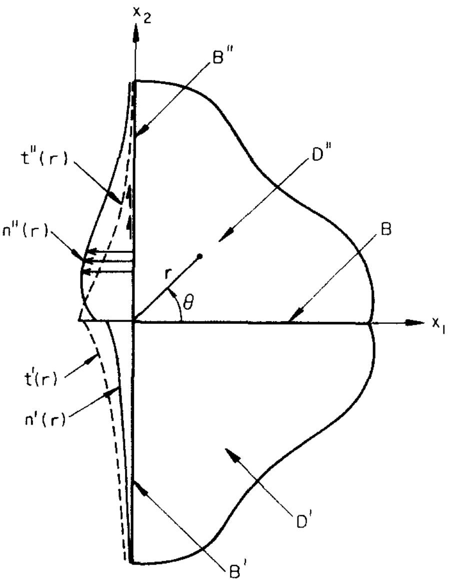

FIG. 1. Bonded quarter-planes under normal and shear loading.

applied loads, and the complete expressions for the stress components must be examined in studying this case. Due to the immense algebraic complexity of these expressions in terms of the three material parameters it was not feasible to include such results in [1].

In a very illuminating discussion of [1] Dundurs [2] pointed out that the problem was such that only *two* combinations of the four elastic constants were sufficient to describe the dependence of its stress field on the material parameters. Defining two composite parameters α, β in terms of three particular combinations k_1, k_2, k_3 used in [1], Dundurs was able to display the dependence of the power λ (determining the order $r^{-\lambda}$ of the singularity) given in [1] very conveniently in an α–β parameter plane. In particular, he outlined the bounded regions (parallelograms) in the α–β plane corresponding to all possible admissible values of the elastic constants μ', μ'', ν', ν'' and exhibited the graphical results in [1] in terms of curves of constant λ. He also pointed out a certain physical significance of the lines in the α–β plane that bound the regions in which the λ-lines are defined.

The simplification resulting from the reduction from three to two in the number of material parameters involved in the stress field has proved to be quite significant, and many further investigations, which as a result of this simplification have become feasible, are included here.

In Section 2 of the present paper, after listing the relation to Dundurs' composite parameters to the parameters used in [1], we recall from Dundurs' discussion [2] the graphical technique for easy location of the point in the α–β plane corresponding to a given set μ', μ'', ν', ν''. Additional refinements are made in locating the polygon in the α–β plane corresponding to μ'/μ'' as well as in exhibiting the dependence on ν', ν'' of the location of the point in this polygon.

In Section 3 the complete integral representations are given for the stress components τ_{rr}, $\tau_{r\theta}$, $\tau_{\theta\theta}$ on each quarter-plane. These expressions are obtained directly from the results presented in [1], but are expressed here explicitly in terms of the parameters α,β. The integral representations appear first in the form of line integrals in the complex transform domain and later, in Section 6, in the corresponding real integral form, since the former are more useful than the latter for investigating the asymptotic behavior and for deriving certain desired global results, whereas the latter are needed to compute stress as a function of position.

Next, in Section 4, the asymptotic behavior of the stress field is given for all physically relevant values of α, β. Depending on the location of the point in the α–β plane, determined by the two materials, the singularity is of order $r^{-\lambda}$, $\log r$, or 1.

Following a suggestion in [2] the "fraction of load transmitted to the foundation at infinity" by each quarter-plane (i.e. the fraction of load transmitted from one quarter-plane to the other) is derived in Section 5. This is a global result and is obtained much more readily from the complex integral representations than it could be from the corresponding real integrals. The answer is a concise, explicit, elementary, algebraic expression that depends only on one composite parameter, α.

Finally, in Section 6 a particular loading, the concentrated normal force, is considered. The real integrals are obtained for the stress components along the bonded edge and they are evaluated numerically for various values of α, β.

2. THE TWO COMPOSITE PARAMETERS α, β

The problem considered in [1] was the two-dimensional one (plane strain or generalized plane stress) of arbitrary integrable tractions applied to the boundary of bonded quarter-planes of dissimilar isotropic elastic materials as depicted in Fig. 1. Primes and double primes are used to denote quantities defined on the lower and upper quarter-planes, respectively. The stress fields τ'_{rr}, $\tau'_{\theta\theta}$, $\tau'_{r\theta}$ on D' and τ''_{rr}, $\tau''_{\theta\theta}$, $\tau''_{r\theta}$ on D'' were given in terms of line integrals in the complex s-plane by equations (19–26) in [1] and are of the form

$$\begin{aligned} \tau'_{rr}(r,\theta) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{\tau}'_{rr}(s,\theta) r^{-(s+2)}\,ds, \text{ etc.}, && \left(0 < r < \infty, -\frac{\pi}{2} < \theta < 0\right), \\ \tau''_{rr}(r,\theta) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{\tau}''_{rr}(s,\theta) r^{-(s+2)}\,ds, \text{ etc.}, && \left(0 < r < \infty, 0 < \theta < \frac{\pi}{2}\right), \end{aligned} \tag{2.1}$$

where the path of integration $\mathrm{Re}(s) = c$ lies in the vertical strip $\mathrm{Re}(s_1) < \mathrm{Re}(s) < -1$. The point denoted by s_1 is the pole in the s-plane of the integrands in (2.1) that is nearest to the vertical line $\mathrm{Re}(s) = -1$ and lies in the open strip $-2 < \mathrm{Re}(s) < -1$; it coincides with a zero of $\Delta(s)$ defined by

$$\Delta(s) = [(k_1-k_2)\cos^2(s\pi/2) - k_1(s+1)^2]^2 + k_3^2\cos^2(s\pi/2)\sin^2(s\pi/2) - k_2^2(s+1)^2, \tag{2.2}$$

where

$$k_1 = 2(k-1), \qquad k_2 = km''-m', \qquad k_3 = km''+m', \tag{2.3}$$

with k, m', m'' related to the shear moduli μ', μ'' and the Poisson's ratios ν', ν'' by

$$k = \mu'/\mu'', \qquad m = \begin{cases} 4(1-\nu) & \text{for plane strain} \\ \dfrac{4}{1+\nu} & \text{for generalized plane stress.} \end{cases} \tag{2.4}$$

This zero s_1 of $\Delta(s)$ occurs on the real axis; its location depends on the elastic constants and is shown in Fig. 3 of [1] for various combinations of k, ν', ν'' in the case of generalized plane stress.

Although it is obvious from (2.2) that the locations of the zeros of $\Delta(s)$ depend only on two parameters, k_1/k_3, k_2/k_3, it is by no means obvious that this is true of the quantities $\hat{\tau}'_{rr}(s, \theta)$ etc., as derived from equations (19)–(26) of [1] and appearing in the integrands in (2.1). That this must be the case was pointed out by Dundurs in [2] on the basis of some general theorems proved by Dundurs in [3]. That is, for certain restrictions on the loading and connectivity of the regions, the stress field in a composite of two isotropic elastic materials depends on only two composite material parameters, which were denoted by α, β in [2] and are related to k_1, k_2, k_3 and hence by (2.3) to k, m', m'' through

$$\alpha = \frac{k_2}{k_3} = \frac{km'' - m'}{km'' + m'}, \qquad \beta = \frac{k_2 - k_1}{k_3} = \frac{k(m''-2)-(m'-2)}{km''+m'}. \tag{2.5}$$

For

$$0 < \nu', \nu'' < \tfrac{1}{2}, \qquad 0 < \mu', \mu'' < \infty, \tag{2.6}$$

the possible values of α, β are contained in parallelograms in the α–β plane as shown in Fig. 2. The parallelograms are bounded by the lines

$$\alpha = \pm 1, \qquad \beta = \begin{cases} (\alpha \pm 1)/4 & \text{for plane strain} \\ (3\alpha \pm 1)/8 & \text{for generalized plane stress.} \end{cases} \tag{2.7}$$

Thus the generalized plane stress parallelogram (dashed lines) is completely contained in the one for plane strain.

Given the four elastic constants μ', μ'', ν', ν'' the composite parameters α, β are uniquely determined by, and easily computed from (2.4) and (2.5). However, one point in the α–β plane may correspond to an infinite number of material combinations; yet certain lines

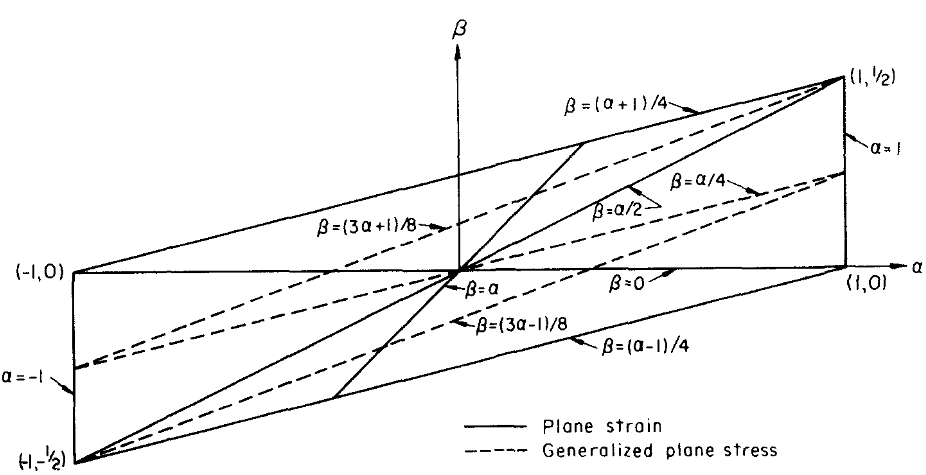

FIG. 2. Parallelograms of physically relevant values of α and β.

in the α–β plane have easily recognizable physical significance.† Several such lines are shown in Fig. 2; the elastic constants corresponding to these lines are indicated in Fig. 3, where the letter "P" or "G" appears if a distinction needs to be made for plane strain or generalized plane stress. One readily verifies from (2.5) that $\beta = \alpha$ corresponds to materials with the same shear moduli, i.e. $k = 1$; (α, β) equals (0, 0) corresponds to identical materials; and $\alpha \to \pm 1$ as $k \to \infty, 0$. Also ν' increases upwardly from 0 to $\frac{1}{2}$ along the left vertical line of the appropriate parallelogram and ν'' increases downwardly likewise along the right vertical. This variation is non-linear for plane strain but is linear for generalized plane stress. A choice of ν', ν'' determines a line across the parallelogram along which k varies from 0 to ∞, and all such lines for which $\nu' = \nu''$ pass through the origin.

For a particular choice of the ratio of shear moduli k, the possible points in the α–β plane are restricted to an easily defined quadrilateral "k-polygon" (degenerate to a straight line for $k = 0, 1, \infty$). From (2.5) it follows that when k and m'' (or k and m') are fixed the values of α, β are restricted to a straight line in the α–β plane that passes through the point $\alpha = \beta = -1$ (or $\alpha = \beta = 1$). The k-polygon can therefore be quickly outlined graphically by connecting these points to the terminal points of the vertical line $\alpha = (k-1)/(k+1)$ representing equal values of ν', ν''. Figure 3 shows a typical k-polygon‡ for $0 < k < 1$.

On each side of the k-polygon the value of one of ν', ν'' is fixed at 0 or $\frac{1}{2}$ while the other varies along that side from 0 to $\frac{1}{2}$. We can easily construct a $\nu' - \nu''$ grid on the k-polygon as follows. Locate the point on the vertical line $\alpha = (k-1)/(k+1)$ corresponding to chosen values of $\nu' = \nu''$. Then connect this point with each of the points (α, β) equals $(-1, -1)$,

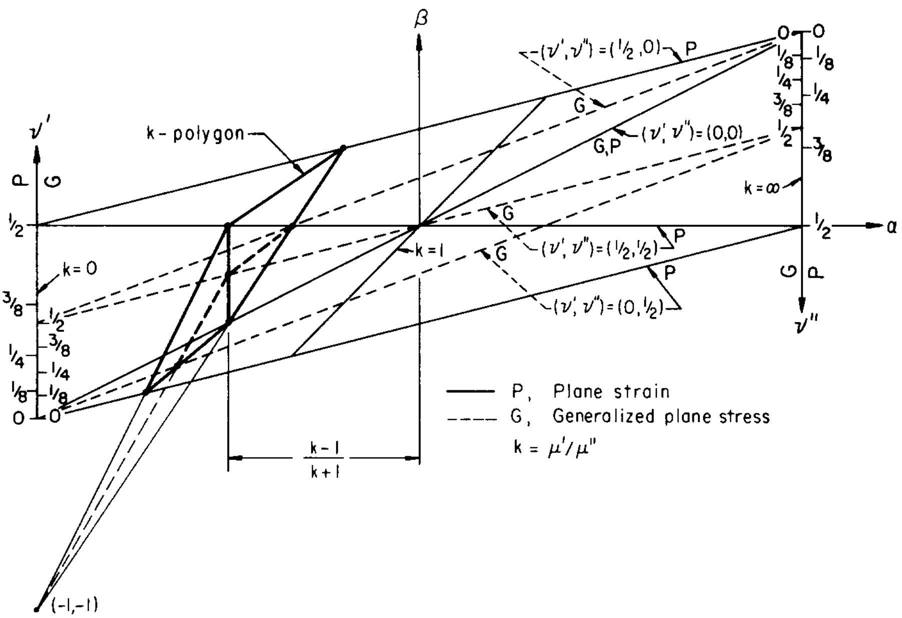

FIG. 3. Elastic-constant values on the lines shown in Fig. 2 and a typical k-polygon ($k = \frac{1}{3}$).

† See Dundurs [2] for further discussion of this significance.

‡ The k-polygon for $k = a$, $a > 1$ results from an inversion through the origin of the k-polygon for $k = 1/a$ since interchanging the two materials merely changes the signs of α, β.

(1, 1) and draw the lines across the polygon. Figure 4 shows an expanded view of the k-polygon in Fig. 3 with such a grid. In a glance one sees approximately where the point corresponding to chosen values of v', v'' lies in the k-polygon.

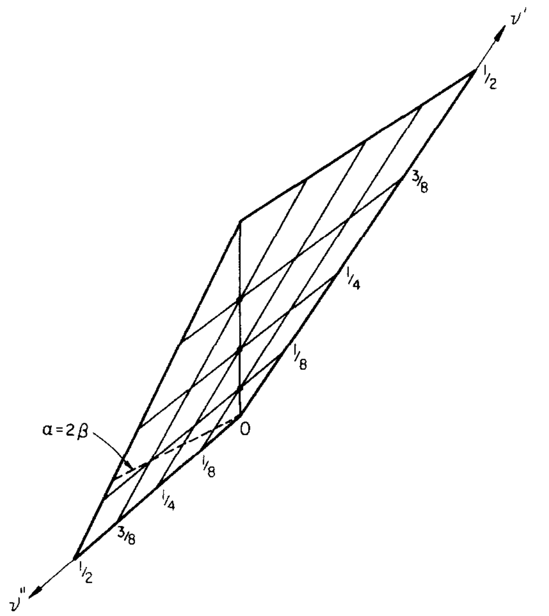

FIG. 4. A typical k-polygon ($k = \frac{1}{3}$, plane strain) with $v' - v''$ grid.

3. THE STRESS FIELD IN TERMS OF α, β

In order to complete the integral representation of the stress field on the two quarter-planes D', D'' given in (2.1) we need only list the expressions $\hat{\tau}'_{rr}(s, \theta)$, $\hat{\tau}''_{rr}(s, \theta)$, etc. As stated previously, these functions follow directly from equations (12), (19)–(23) in [1] in terms of the three parameters k, m', m''. Making use of (2.5) we can write these expressions in terms of α, β; the transition is effected as follows. First define $\mathscr{D}(\alpha, \beta; s)$, $\mathscr{C}_i(\alpha, \beta; s, \theta)$ by

$$\begin{aligned} \frac{1}{(1-\alpha)^2}\mathscr{D}(\alpha, \beta; s) &= \frac{1}{4(m')^2}\Delta(k, m', m''; s), \\ \frac{1}{(1-\alpha)^2}\mathscr{C}_i(\alpha, \beta; s, \theta) &= -\frac{4}{(m'\mu'')^2}c_i(\mu', m', \mu'', m''; s, \theta), \qquad (i = 1, 2, 3, 4), \end{aligned} \tag{3.1}$$

where Δ is given by (2.2) and c'_i, c''_i by equations (23) of [1], so that, in view of (2.5),

$$\mathscr{D}(\alpha, \beta; s) = [\beta\cos^2(s\pi/2) + (\alpha-\beta)(s+1)^2]^2 + \cos^2(s\pi/2)\sin^2(s\pi/2) - \alpha^2(s+1)^2, \tag{3.2}$$

and $\mathscr{C}''_i$, $\mathscr{C}'_i$ are related by

$$\begin{aligned} \mathscr{C}''_1(\alpha, \beta; s, \theta) &= \mathscr{C}'_3(-\alpha, -\beta; s, -\theta), \qquad & \mathscr{C}''_2(\alpha, \beta; s, \theta) &= -\mathscr{C}'_4(-\alpha, -\beta; s, -\theta), \\ \mathscr{C}''_3(\alpha, \beta; s, \theta) &= \mathscr{C}'_1(-\alpha, -\beta; s, -\theta), \qquad & \mathscr{C}''_4(\alpha, \beta; s, \theta) &= -\mathscr{C}'_2(-\alpha, -\beta; s, -\theta). \end{aligned} \tag{3.3}$$

Equations (20) in [1], giving the transforms of the Airy functions, now go over to

$$\hat{\phi}'(\alpha, \beta; s, \theta) = -\tfrac{1}{16} \sum_{i=1}^{4} \mathscr{C}_i'(\alpha, \beta; s, \theta) L_i(s)/\mathscr{D}(\alpha, \beta; s),$$
$$\hat{\phi}''(\alpha, \beta; s, \theta) = -\tfrac{1}{16} \sum_{i=1}^{4} \mathscr{C}_i''(\alpha, \beta; s, \theta) L_i(s)/\mathscr{D}(\alpha, \beta; s), \qquad (3.4)$$

where, as in (19) of [1],

$$L_1(s) = \frac{\hat{n}'(s)}{s(s+1)}, \qquad L_2(s) = \frac{\hat{t}'(s)}{s(s+1)}, \qquad L_3(s) = \frac{\hat{n}''(s)}{s(s+1)}, \qquad L_4(s) = \frac{\hat{t}''(s)}{s(s+1)}, \qquad (3.5)$$

and $\hat{n}'(s)$, $\hat{t}'(s)$, $\hat{n}''(s)$, $\hat{t}''(s)$ stand for the transforms of the applied normal and shear tractions, i.e.

$$\hat{n}'(s) = \int_0^\infty n'(r) r^{s+1}\, dr, \text{ etc.} \qquad (3.6)$$

In order to list the functions $\mathscr{C}_i'$ concisely we introduce some abbreviated notation. Let $c^{(k)}$, $s^{(k)}$ $(k = 1, 2, 3, 4)$ be defined through

$$\begin{aligned} c^{(1)}(s, \theta) &= \cos[s(\pi/2+\theta)], & s^{(1)}(s, \theta) &= \sin[s(\pi/2+\theta)], \\ c^{(2)}(s, \theta) &= \cos[s(\pi/2-\theta)], & s^{(2)}(s, \theta) &= \sin[s(\pi/2-\theta)], \\ c^{(3)}(s, \theta) &= \cos[s(\pi/2+\theta)+2\theta], & s^{(3)}(s, \theta) &= \sin[s(\pi/2+\theta)+2\theta], \\ c^{(4)}(s, \theta) &= \cos[s(\pi/2-\theta)-2\theta], & s^{(4)}(s, \theta) &= \sin[s(\pi/2-\theta)-2\theta]. \end{aligned} \qquad (3.7)$$

Then $\mathscr{C}_i'$, as obtained from (2.5), (3.1) and (23) of [1], appear as

$$\mathscr{C}_i'(\alpha, \beta; s, \theta) = \sum_{k=1}^{4} [P_i^{(k)}(\alpha, \beta; s) + Q_i^{(k)}(\alpha, \beta; s)\cos^2(s\pi/2)] c^{(k)}(s, \theta), \qquad (i = 1, 3),$$
$$\mathscr{C}_i'(\alpha, \beta; s, \theta) = \sum_{k=1}^{4} [P_i^{(k)}(\alpha, \beta; s) + Q_i^{(k)}(\alpha, \beta; s)\cos^2(s\pi/2)] s^{(k)}(s, \theta), \qquad (i = 2, 4), \qquad (3.8)$$

in which the thirty-two functions $P_i^{(k)}$, $Q_i^{(k)}$, $(i, k = 1, 2, 3, 4)$ are defined by

$$\begin{aligned} P_1^{(1)}(\alpha, \beta; s) &= -4(\alpha-\beta)^2(2s^4+9s^3+14s^2+9s+2)+4\alpha^2(2s^2+5s+2) \\ &\quad +8(\alpha-\beta)(1-\alpha)(s^2+2s+1)-2\alpha(1-\alpha)(2s^2+3s+2)-2(1-\alpha^2)(s+2), \\ Q_1^{(1)}(\alpha, \beta; s) &= -4(\alpha-\beta)(1+\beta)(2s^2+5s+2)+8\beta(1-\alpha), \\ P_1^{(2)}(\alpha, \beta; s) &= 4(\alpha-\beta)(1-\beta)(s^3+4s^2+5s+2)-2(1+\alpha)(s+2), \\ Q_1^{(2)}(\alpha, \beta; s) &= -4(\beta^2-1)(s+2), \\ P_1^{(3)}(\alpha, \beta; s) &= 4(\alpha-\beta)^2(2s^4+7s^3+8s^2+3s)-2\alpha(1+\alpha)(2s^2+s)-2(1+\alpha)^2 s, \\ Q_1^{(3)}(\alpha, \beta; s) &= 4(\alpha-\beta)(1+\beta)(2s^2+3s), \\ P_1^{(4)}(\alpha, \beta; s) &= 4(\alpha-\beta)(1-\beta)(s^3+2s^2+s)-2(1+\alpha)s, \\ Q_1^{(4)}(\alpha, \beta; s) &= -4(\beta^2-1)s, \\ P_2^{(1)}(\alpha, \beta; s) &= -4(\alpha-\beta)^2(2s^4+7s^3+8s^2+3s)-8\beta(1-\alpha)(s^2+2s+1) \\ &\quad +2\alpha(1+\alpha)(2s^2+3s)+2(1+\alpha)(1-\alpha)s, \end{aligned} \qquad (3.9)$$

$$
\begin{aligned}
Q_2^{(1)}(\alpha, \beta; s) &= -4(\alpha-\beta)(1+\beta)(2s^2+3s)+8\beta(1-\alpha), \\
P_2^{(2)}(\alpha, \beta; s) &= 4(\alpha-\beta)(1-\beta)(s^3+2s^2+s)-2(1+\alpha)s, \\
Q_2^{(2)}(\alpha, \beta; s) &= 4(1-\beta^2)s, \\
P_2^{(3)}(\alpha, \beta; s) &= 4(\alpha-\beta)^2(2s^4+5s^3+4s^2+s)-2\alpha(1+\alpha)(2s^2+s)+2(1-\alpha^2)s, \\
Q_2^{(3)}(\alpha, \beta; s) &= 4(\alpha-\beta)(1+\beta)(2s^2+s), \\
P_2^{(4)}(\alpha, \beta; s) &= 4(\alpha-\beta)(1-\beta)(s^3+2s^2+s)-2(1+\alpha)s, \\
Q_2^{(4)}(\alpha, \beta; s) &= 4(1-\beta^2)s, \\
\frac{1}{1+\alpha}P_3^{(1)}(\alpha, \beta; s) &= -4(\alpha-\beta)(s^3+3s^2+4s+2)+8\alpha(s+1)-2(s+2), \\
\frac{1}{1+\alpha}Q_3^{(1)}(\alpha, \beta; s) &= -8\beta(s+1)+4(1-\beta)(s+2), \\
\frac{1}{1+\alpha}P_3^{(2)}(\alpha, \beta; s) &= 4\beta(s^2+3s+2)-2(s+2), \qquad Q_3^{(2)} = 0, \\
\frac{1}{1+\alpha}P_3^{(3)}(\alpha, \beta; s) &= -4(\alpha-\beta)(s^3+3s^2+2s)-2s, \\
\frac{1}{1+\alpha}Q_3^{(3)}(\alpha, \beta; s) &= 4(1+\beta)s \\
\frac{1}{1+\alpha}P_3^{(4)}(\alpha, \beta; s) &= 4\beta(s^2+s)-2s, \qquad Q_3^{(4)} = 0, \\
\frac{1}{1+\alpha}P_4^{(1)}(\alpha, \beta; s) &= 4(\alpha-\beta)(s^3+3s^2+4s+2)-8\alpha(s+1)+2s, \\
\frac{1}{1+\alpha}Q_4^{(1)}(\alpha, \beta; s) &= 8\beta(s+1)-4(1-\beta)s, \\
\frac{1}{1+\alpha}P_4^{(2)}(\alpha, \beta; s) &= -4\beta(s^2+s)-2s, \qquad Q_4^{(2)} = 0, \\
\frac{1}{1+\alpha}P_4^{(3)}(\alpha, \beta; s) &= 4(\alpha-\beta)(s^3+s^2)+2s, \\
\frac{1}{1+\alpha}Q_4^{(3)}(\alpha, \beta; s) &= -4(1+\beta)s \\
\frac{1}{1+\alpha}P_4^{(4)}(\alpha, \beta; s) &= -4\beta(s^2+s)-2s, \qquad Q_4^{(4)} = 0.
\end{aligned}
\tag{3.9 (cont.)}
$$

Equations (3.3), (3.7), (3.8) and (3.9) define $\mathscr{C}'_i$, $\mathscr{C}''_i$ explicitly in terms of α, β. Using the relations (12) of [1],

$$
\begin{aligned}
\hat{\tau}_{rr}(s, \theta) &= \left(\frac{d^2}{d\theta^2}-s\right)\hat{\phi}(s, \theta), \qquad \hat{\tau}_{\theta\theta}(s, \theta) = s(s+1)\hat{\phi}(s, \theta), \\
\hat{\tau}_{r\theta}(s, \theta) &= (s+1)\frac{d}{d\theta}\hat{\phi}(s, \theta),
\end{aligned}
\tag{3.10}
$$

in conjunction with (3.4), (3.7), (3.8) and (3.9), we obtain for the integrands in (2.1)

$$\begin{aligned}
\hat{\tau}'_{rr}(s,\theta) &= \frac{s+1}{16\mathscr{D}(s)}\left(s\sum_{k=1}^{4}+4\sum_{k=3,4}\right)\left(c^{(k)}(s,\theta)\sum_{i=1,3}+s^{(k)}(s,\theta)\sum_{i=2,4}\right)M_i^{(k)}(s)L_i(s),\\
\hat{\tau}'_{\theta\theta}(s,\theta) &= -\frac{s(s+1)}{16\mathscr{D}(s)}\sum_{k=1}^{4}\left(c^{(k)}(s,\theta)\sum_{i=1,3}+s^{(k)}(s,\theta)\sum_{i=2,4}\right)M_i^{(k)}(s)L_i(s),\\
\hat{\tau}'_{r\theta}(s,\theta) &= -\frac{s+1}{16\mathscr{D}(s)}\left(s\sum_{k=1}^{4}+2\sum_{k=3,4}\right)\left(s^{(k)}(s,\theta)\sum_{i=1,3}-c^{(k)}(s,\theta)\sum_{i=2,4}\right)(-1)^kM_i^{(k)}(s)L_i(s),
\end{aligned}\tag{3.11}$$

for $-\pi/2<\theta<0$ and

$$\begin{aligned}
\hat{\tau}''_{rr}(s,\theta) &= \frac{s+1}{16\mathscr{D}(s)}\left(s\sum_{k=1}^{4}+4\sum_{k=3,4}\right)\left(c^{(k)}(s,-\theta)\sum_{i=1,3}-s^{(k)}(s,-\theta)\sum_{i=2,4}\right)\bar{M}_{i+2}^{(k)}(s)L_i(s),\\
\hat{\tau}''_{\theta\theta}(s,\theta) &= -\frac{s(s+1)}{16\mathscr{D}(s)}\sum_{k=1}^{4}\left(c^{(k)}(s,-\theta)\sum_{i=1,3}-s^{(k)}(s,-\theta)\sum_{i=2,4}\right)\bar{M}_{i+2}^{(k)}(s)L_i(s),\\
\hat{\tau}''_{r\theta}(s,\theta) &= \frac{s+1}{16\mathscr{D}(s)}\left(s\sum_{k=1}^{4}+2\sum_{k=3,4}\right)\left(s^{(k)}(s,-\theta)\sum_{i=1,3}+c^{(k)}(s,-\theta)\sum_{i=2,4}\right)(-1)^k\bar{M}_{i+2}^{(k)}(s)L_i(s),
\end{aligned}\tag{3.12}$$

for $0<\theta<\pi/2$, in which

$$\begin{aligned}
M_i^{(k)}(s) &= P_i^{(k)}(s)+Q_i^{(k)}(s)\cos^2(s\pi/2),\\
\bar{M}_{i+2}^{(k)}(s) &= \bar{P}_{i+2}^{(k)}(s)+\bar{Q}_{i+2}^{(k)}(s)\cos^2(s\pi/2),\\
\bar{P}_i^{(k)}(\alpha,\beta;s) = P_i^{(k)}(-\alpha,-\beta;s),&\qquad \bar{Q}_i^{(k)}(\alpha,\beta;s) = Q_i^{(k)}(-\alpha,-\beta;s),\quad (i,k=1,2,3,4),\\
\bar{P}_5^{(k)}=\bar{P}_1^{(k)},\quad \bar{P}_6^{(k)}=\bar{P}_2^{(k)},&\quad \bar{Q}_5^{(k)}=\bar{Q}_1^{(k)},\quad \bar{Q}_6^{(k)}=\bar{Q}_2^{(k)},\quad (k=1,2,3,4).
\end{aligned}\tag{3.13}$$

Equations (3.2), (3.5)–(3.7), (3.9), (3.11)–(3.13) with (2.1) give the complete integral representation for the stress field on the two bonded quarter-planes D', D'' in terms of α, β.

The stress components $\tau_{\theta\theta}$, $\tau_{r\theta}$ are continuous at $\theta=0\,[\tau'_{\theta\theta}(r,0^-)=\tau''_{\theta\theta}(r,0^+)$, etc.] while τ_{rr} is, in general, discontinuous there. In evaluating these quantities at $\theta=0^+$ or $\theta=0^-$ (as $\theta\to 0$ from $\theta>0$ or $\theta<0$) we can set $\theta=0$ in the integrands of (2.1), i.e. in (3.11) and (3.12). The numerical computation of the resulting integrals for various values of α, β is one of the usual means for exhibiting the effects of the composite. The integrands appropriate to such computations are recorded here for later use.

$$\begin{aligned}
\hat{\tau}'_{rr}(\alpha,\beta;s,0^-) &= \sum_{i=1}^{4}F_i(\alpha,\beta;s)L_i(s)/2\mathscr{D}(s),\\
\hat{\tau}''_{rr}(\alpha,\beta;s,0^+) &= \sum_{i=1}^{4}\bar{F}_{i+2}(\alpha,\beta;s)L_i(s)/2\mathscr{D}(s),\\
\hat{\tau}'_{\theta\theta}(\alpha,\beta;s,0^-) = \hat{\tau}''_{\theta\theta}(\alpha,\beta;s,0^+) &= \sum_{i=1}^{4}G_i(\alpha,\beta;s)L_i(s)/2\mathscr{D}(s),\\
\hat{\tau}'_{r\theta}(\alpha,\beta;s,0^-) = \hat{\tau}''_{r\theta}(\alpha,\beta;s,0^+) &= \sum_{i=1}^{4}H_i(\alpha,\beta;s)L_i(s)/2\mathscr{D}(s),
\end{aligned}\tag{3.14}$$

where

$$
\begin{aligned}
F_1(\alpha,\beta;s) &= s(s+1)\{(\alpha-\beta)(3\alpha-4\beta+1)(s+1)^3+[(\alpha-\beta)(\alpha-4\beta+3)-\alpha(1-\alpha)](s+1)^2\\
&\quad -(1+\alpha)^2(s+1)-2(1+\alpha)+[(1+\beta)(3\alpha-4\beta+1)(s+1)+(1+\beta)(\alpha-4\beta+3)\\
&\quad -(1-\alpha)]\cos^2(s\pi/2)\}\cos(s\pi/2),\\
F_2(\alpha,\beta;s) &= s(s+1)\{(\alpha-\beta)(3\alpha-4\beta+1)(s+1)^3+(1-\alpha)(\alpha-2\beta)(s+1)^2-2\alpha(1+\alpha)(s+1)\\
&\quad +[(1+\beta)(3\alpha-4\beta+1)(s+1)+(1-\alpha)(1+2\beta)]\cos^2(s\pi/2)\}\sin(s\pi/2),\\
F_3(\alpha,\beta;s) &= s(s+1)\{-(\alpha-\beta)(s+1)^3+(3\beta-2\alpha)(s+1)^2+(4\beta-\alpha-1)(s+1)-2\\
&\quad +[(1-\beta)(s+1)+\beta+2]\cos^2(s\pi/2)\}(1+\alpha)\cos(s\pi/2),\\
F_4(\alpha,\beta;s) &= s(s+1)\{(\alpha-\beta)(s+1)^3+(\alpha-2\beta)(s+1)^2-2\alpha(s+1)\\
&\quad -[(1-\beta)(s+1)+1+2\beta]\cos^2(s\pi/2)\}(1+\alpha)\sin(s\pi/2),\\
\bar{F}_i(\alpha,\beta;s) &= (-1)^{i+1}F_i(-\alpha,-\beta;s),\qquad (i=1,2,3,4),\qquad \bar{F}_5=\bar{F}_1,\quad \bar{F}_6=\bar{F}_2,\\
G_1(\alpha,\beta;s) &= s(s+1)\{-(\alpha-\beta)(s+1)^3+\beta(s+1)^2+(1+\alpha)(s+1)\\
&\quad -[(1+\beta)(s+1)+\beta]\cos^2(s\pi/2)\}(1-\alpha)\cos(s\pi/2),\\
G_2(\alpha,\beta;s) &= s(s+1)\{-(\alpha-\beta)(s+1)^3+\alpha(s+1)^2\\
&\quad +[-(1+\beta)(s+1)+1]\cos^2(s\pi/2)\}(1-\alpha)\sin(s\pi/2),\\
G_3(\alpha,\beta;s) &= G_1(-\alpha,-\beta;s),\qquad G_4(\alpha,\beta;s) = -G_2(-\alpha,-\beta;s),\\
H_1(\alpha,\beta;s) &= s(s+1)\{-(\alpha-\beta)(s+1)^3-\alpha(s+1)^2\\
&\quad -[(1+\beta)(s+1)+1]\cos^2(s\pi/2)\}(1-\alpha)\sin(s\pi/2),\\
H_2(\alpha,\beta;s) &= s(s+1)\{(\alpha-\beta)(s+1)^3+\beta(s+1)^2-(1+\alpha)(s+1)\\
&\quad +[(1+\beta)(s+1)-\beta]\cos^2(s\pi/2)\}(1-\alpha)\cos(s\pi/2),\\
H_3(\alpha,\beta;s) &= -H_1(-\alpha,-\beta;s),\qquad H_4(\alpha,\beta;s) = H_2(-\alpha,-\beta;s).
\end{aligned}
\tag{3.15}
$$

4. THE STRESS SINGULARITY

In this section the asymptotic behavior as $r \to 0$ of the stress field is investigated. The results are obtained for all physically relevant values of the composite parameters α, β. In [1] the asymptotic investigations were limited to the study of the dependence of the power λ on the elastic constants for the case when the stress components are of order $r^{-\lambda}$ as $r \to 0$ with $\lambda > 0$. Here the asymptotic behavior is given, including its dependence on θ and α, β, for the three separate cases described by Dundurs [2] and characterized by $\alpha(\alpha-2\beta) \gtreqless 0$.

Case I. $\alpha(\alpha-2\beta) > 0$

This is the case mentioned previously and studied in [1], and is distinguished by the occurrence of a zero of $\mathscr{D}(\alpha, \beta; s)$, as given in (3.2), in the vertical open strip $-2 < \mathrm{Re}(s) <$

-1; we denote the location of this zero by s_1. It was found in [1] to occur always on the real axis ($s_1 = \sigma_1$ is real), which precludes the possibility of an oscillatory singular behavior.

According to (35) of [1], the asymptotic behavior of the stress field, when $\alpha(\alpha-2\beta) > 0$, is given by

$$\tau'_{rr}(r,\theta) = r^{-\lambda} \lim_{s\to s_1} (s-s_1)\hat{\tau}'_{rr}(s,\theta) + o(r^{-\lambda}), \text{ etc.}, \qquad -\frac{\pi}{2} < \theta < 0,$$
$$\tau''_{rr}(r,\theta) = r^{-\lambda} \lim_{s\to s_1} (s-s_1)\hat{\tau}''_{rr}(s,\theta) + o(r^{-\lambda}), \text{ etc.}, \qquad 0 < \theta < \frac{\pi}{2}, \tag{4.1}$$

as $r \to 0$, where $\hat{\tau}'_{rr}(s,\theta)$, $\hat{\tau}''_{rr}(s,\theta)$, etc., are given in (3.11), (3.12) and

$$\lambda = \lambda(\alpha,\beta) = \sigma_1 + 2, \qquad 0 < \lambda < 1. \tag{4.2}$$

The λ-lines (lines of constant λ) are the hyperbolae in the α–β plane satisfying the equation

$$\begin{aligned}&[\cos^2(\lambda\pi/2)-(1-\lambda)^2]^2\beta^2 + 2(1-\lambda)^2[\cos^2(\lambda\pi/2)-(1-\lambda)^2]\alpha\beta \\ &\quad + (1-\lambda)^2[(1-\lambda)^2-1]\alpha^2 + \cos^2(\lambda\pi/2)\sin^2(\lambda\pi/2) \\ &= \mathscr{D}(\alpha,\beta;\lambda-2) = 0.\end{aligned} \tag{4.3}$$

These lines are shown in Fig. 5 for values of λ appropriate to the regions† $\alpha(\alpha-2\beta) > 0$ over the entire plane-strain parallelogram, which, it should be recalled, includes the generalized plane stress parallelogram. These lines coincide with those shown in [2], as obtained from [1], on the generalized plane stress parallelogram. The largest value of λ is 0·41 and occurs in plane strain when one material is rigid and the other is incompressible, i.e. at $k = 0$, $\nu' = \frac{1}{2}$ and $k = \infty$, $\nu'' = \frac{1}{2}$.

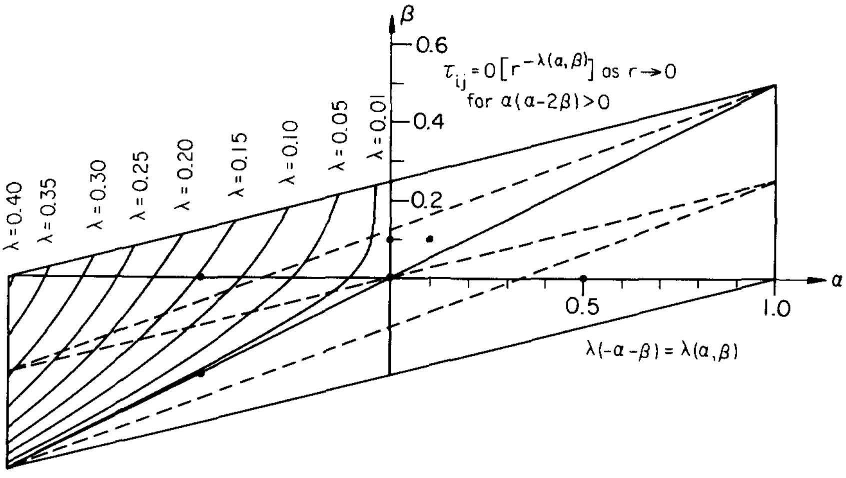

FIG. 5. Dependence of the order of the stress singularity on α, β when $\alpha(\alpha-2\beta) > 0$.

† There are two regions $\alpha(\alpha-2\beta) > 0$ corresponding to $k \lessgtr 1$. The λ-lines for $k > 1$ are obtained from those for $k < 1$ by a reflection through the origin in the α–β plane.

Case II. $\alpha(\alpha-2\beta) = 0$

This condition corresponds to equation (50) of [1], $k_2(k_2 - 2k_1) = 0$, as is readily verified by the use of (2.5), and represents those combinations of materials for which $\mathscr{D}(\alpha, \beta; s)$ has no zero in $-2 < \text{Re}(s) < -1$ but $s = -2$ is a zero of order two. A singularity of order $r^{-\lambda}$, for any $\lambda > 0$, cannot occur in the stress field. Furthermore, as will be revealed shortly, the stress singularity depends on the applied tractions $n'(r), t'(r), n''(r), t''(r)$ primarily through their values at $r = 0$. In order to obtain the asymptotic behavior in this case the integrands in (2.1) must be considered in detail. The decrease in their complexity resulting from the reduction in the number of material parameters from three to two has made this feasible.

Suppose the load functions $n'(r)$, $t'(r)$, $n''(r)$, $t''(r)$ are continuously differentiable and absolutely integrable on $0 \leq r < \infty$. Then the transforms $\hat{n}'(s)$, $\hat{t}'(s)$, etc., satisfy†

$$\hat{n}'(s) = \frac{n'(0)}{s+2} + N'(s), \qquad \hat{t}'(s) = \frac{t'(0)}{s+2} + T'(s), \text{etc.} \qquad [-3 < \text{Re}(s) < -1], \tag{4.4}$$

where $N'(s)$, $T'(s)$ etc., are analytic in $-3 < \text{Re}(s) < -1$. Thus the transforms of the load functions have a simple pole at $s = -2$ when $n'(0)$, $t'(0)$ etc. are non-zero and are analytic elsewhere in the open strip $-3 < \text{Re}(s) < -1$. An examination of $\hat{\tau}'_{rr}(s, \theta)$, $\hat{\tau}''_{rr}(s, \theta)$, etc. given in (3.11) and (3.12) reveals that the numerators of the coefficients of $\hat{n}'(s)$, $\hat{t}'(s)$, etc., have simple zeros at $s = -2$. Therefore, and since $\mathscr{D}(s)$ in the denominators has a zero of order two at $s = -2$ for $\alpha(\alpha - 2\beta) = 0$, it follows that the functions $\hat{\tau}'_{rr}(s, \theta)$, $\hat{\tau}''_{rr}(s, \theta)$, etc. in the integrands of (2.1) have poles of order two at $s = -2$ when $n'(0)$, $t'(0)$, etc. are non-zero. Furthermore, one can verify that these functions are analytic in a vertical open strip containing $s = -2$ and that all conditions are satisfied that are necessary‡ for obtaining the asymptotic behavior of (2.1) from the residue of the integrands at $s = -2$. Carrying out these residue computations appropriate to a pole of order two at $s = -2$ we obtain, in terms of the parameter combinations

$$\begin{aligned} A(\alpha, \beta) &= -\frac{2\alpha(1-2\beta)}{16\beta^2 + \pi^2(1-\alpha^2)}, \qquad B(\alpha, \beta) = \frac{(1+\alpha)(1-2\beta)}{16\beta^2 + \pi^2(1-\alpha^2)} \\ C(\alpha, \beta) &= \frac{2(\alpha-2\beta)}{16\beta^2 + \pi^2(1-\alpha^2)}, \qquad \bar{A} = A(-\alpha, -\beta), \qquad \bar{B} = B(-\alpha, -\beta) \\ \bar{C} &= C(-\alpha, -\beta), \end{aligned} \tag{4.5}$$

the following asymptotic behavior as $r \to 0$ when $\alpha(\alpha - 2\beta) = 0$.

$$\begin{aligned} \tau'_{rr}(r, \theta) &= [An'(0) + \bar{A}n''(0)][1 - \cos(2\theta)] \log r + \{C[\sin(2\theta) - 2\theta] \\ &\quad + \pi[\bar{B} - B\cos(2\theta)]\}[t'(0) - t''(0)] \log r + O(1), \\ \tau'_{\theta\theta}(r, \theta) &= [An'(0) + \bar{A}n''(0)][1 + \cos(2\theta)] \log r + \{-C[\sin(2\theta) - 2\theta] \\ &\quad + \pi[\bar{B} + B\cos(2\theta)]\}[t'(0) - t''(0)] \log r + O(1), \\ \tau'_{r\theta}(r, \theta) &= [An'(0) + \bar{A}n''(0)] \sin(2\theta) \log r + \{C[1 + \cos(2\theta)] \\ &\quad + \pi B \sin(2\theta)\}[t'(0) - t''(0)] \log r + O(1), \end{aligned} \tag{4.6}$$

† See [4] Vol. 1, p. 59, p. 144 and Vol. 2, Ch. 4, para. 2, Ch. 5, para. 2.

‡ See [5] for more detail on such a computation.

for $-\pi/2 < \theta < 0$ and

$$\begin{aligned}
\tau''_{rr}(r,\theta) &= [An'(0)+\bar{A}n''(0)][1-\cos(2\theta)]\log r+\{-\bar{C}[\sin(2\theta)-2\theta] \\
&\quad +\pi[B-\bar{B}\cos(2\theta)]\}[t'(0)-t''(0)]\log r+O(1), \\
\tau''_{\theta\theta}(r,\theta) &= [An'(0)+\bar{A}n''(0)][1+\cos(2\theta)]\log r+\{\bar{C}[\sin(2\theta)+2\theta] \\
&\quad +\pi[B+\bar{B}\cos(2\theta)]\}[t'(0)-t''(0)]\log r+O(1), \\
\tau''_{r\theta}(r,\theta) &= [An'(0)+\bar{A}n''(0)]\sin(2\theta)\log r+\{C[1+\cos(2\theta)] \\
&\quad +\pi\bar{B}\sin(2\theta)\}[t'(0)-t''(0)]\log r+O(1),
\end{aligned} \tag{4.7}$$

for $0 < \theta < \pi/2$. When $n'(0), t'(0)$, etc., vanish the resulting asymptotic expansions are obtained from (4.6) and (4.7) by replacing $n'(0)\log r$, $t'(0)\log r$, etc., by $-\hat{n}'(-2)$, $-\hat{t}'(-2)$, etc., from (4.4) and $O(1)$ by $o(1)$.

Equations (4.6) and (4.7) are valid only when one of $\alpha = 0$ or $\alpha = 2\beta$ is satisfied. From (4.5) we see that $A, \bar{A}$ vanish when $\alpha = 0$ and $C, \bar{C}$ vanish when $\alpha = 2\beta$. At the point of intersection of these two lines in the α–β plane, i.e. at $(\alpha, \beta) = (0, 0)$, the two materials are identical; $A, \bar{A}, C, \bar{C}$ all vanish while $B, \bar{B}$ both equal $1/\pi^2$; and we recover from (4.6) and (4.7)

$$\begin{aligned}
\tau'_{rr}(r,\theta) = \tau''_{rr}(r,\theta) &= \frac{1}{\pi}[1-\cos(2\theta)][t'(0)-t''(0)]\log r+O(1), \\
\tau'_{\theta\theta}(r,\theta) = \tau''_{\theta\theta}(r,\theta) &= \frac{1}{\pi}[1+\cos(2\theta)][t'(0)-t''(0)]\log r+O(1), \\
\tau'_{r\theta}(r,\theta) = \tau''_{r\theta}(r,\theta) &= \frac{1}{\pi}\sin(2\theta)[t'(0)-t''(0)]\log r+O(1).
\end{aligned} \tag{4.8}$$

This is the well-known logarithmic singularity due to a finite discontinuity in the shear load on a half-plane. We also recover the result that such a discontinuity in the normal load on a half-plane does not produce unbounded stresses.

When $\alpha = 0, \beta \neq 0$, then $A, \bar{A}$ are zero but $C, \bar{C}$ are not and there is no essential difference in the asymptotic behavior in (4.6) and (4.7), and that in (4.8) for identical materials. That is, a finite discontinuity in the applied normal traction at $r = 0$ does not lead to an unbounded stress field, and a logarithmic singularity occurs at $r = 0$ for applied shear tractions only if these tractions are discontinuous at $r = 0$.

When $\alpha = 2\beta \neq 0$, then $C, \bar{C}$ are zero but $A, \bar{A}$ are not and (4.6) and (4.7) reveal a singular behavior that is essentially different from that in (4.8) for identical materials. Here the stress field has a logarithmic singularity at $r = 0$ associated with applied normal tractions except for the one particular discontinuity in these tractions which causes $An'(0)+\bar{A}n''(0)$ to vanish. From (4.4), with $\alpha = 2\beta$ this occurs only when

$$\frac{n'(0)}{n''(0)} = \frac{1+2\beta}{1-2\beta}. \tag{4.9}$$

Thus a continuous normal traction produces a logarithmic singularity in the stress field for all composites satisfying $\alpha = 2\beta$ unless the materials are identical.

Case III. $\alpha(\alpha-2\beta) < 0$

Under this condition there is no zero of $\mathscr{D}(\alpha, \beta; s)$ in $-2 < \mathrm{Re}(s) < -1$ and $s = -2$ is a simple zero. With the same assumptions on the load functions $n'(r)$, $t'(r)$, etc. as made in case II, it follows that $s = -2$ is a simple pole of $\hat{\tau}'_{rr}(s, \theta)$, $\hat{\tau}''_{rr}(s, \theta)$, etc. when $n'(0)$, $t'(0)$, etc. are non-zero. From the residue calculations of the integrals in (2.1) appropriate to a simple pole at $s = -2$ we obtain, in terms of the parameter combinations

$$D(\alpha, \beta) = \frac{1-2\beta}{\alpha-2\beta}, \qquad E(\alpha, \beta) = \frac{1+\alpha}{\alpha-2\beta}$$
$$\bar{D} = D(-\alpha, -\beta), \qquad \bar{E} = E(-\alpha, -\beta) \tag{4.10}$$

the following asymptotic expansions for the components of stress as $r \to 0$ valid for those composites satisfying $\alpha(\alpha-2\beta) < 0$.

$$\begin{aligned}
\tau'_{rr}(r, \theta) &= \tfrac{1}{4}[3-D+(1+D)\cos(2\theta)]n'(0)+\tfrac{1}{4}E[1-\cos(2\theta)]n''(0)-\sin(2\theta)t'(0)\\
&\quad+\frac{1+\alpha}{4\alpha}\left\{\sin(2\theta)-2\theta-\pi+\frac{\pi}{2}D[1-\cos(2\theta)]\right\}[t'(0)-t''(0)]+o(1),\\
\tau'_{\theta\theta}(r, \theta) &= \tfrac{1}{4}[3-D-(1+D)\cos(2\theta)]n'(0)+\tfrac{1}{4}E[1+\cos(2\theta)]n''(0)+\sin(2\theta)t'(0)\\
&\quad+\frac{1+\alpha}{4\alpha}\left\{-\sin(2\theta)-2\theta-\pi+\frac{\pi}{2}D[1+\cos(2\theta)]\right\}[t'(0)-t''(0)]+o(1),\\
\tau'_{r\theta}(r, \theta) &= -\tfrac{1}{4}(1+D)\sin(2\theta)n'(0)+\tfrac{1}{4}E\sin(2\theta)n''(0)-\cos(2\theta)t'(0)\\
&\quad+\frac{1+\alpha}{4\alpha}\left[1+\cos(2\theta)+\frac{\pi}{2}D\sin(2\theta)\right][t'(0)-t''(0)]+o(1),
\end{aligned} \tag{4.11}$$

for $-\pi/2 < \theta < 0$ and

$$\begin{aligned}
\tau''_{rr}(r, \theta) &= \tfrac{1}{4}\bar{E}[1-\cos(2\theta)]n'(0)+\tfrac{1}{4}[3-\bar{D}+(1+\bar{D})\cos(2\theta)]n''(0)-\sin(2\theta)t''(0)\\
&\quad+\frac{1-\alpha}{4\alpha}\left\{\sin(2\theta)-2\theta+\pi-\frac{\pi}{2}\bar{D}[1-\cos(2\theta)]\right\}[t'(0)-t''(0)]+o(1),\\
\tau''_{\theta\theta}(r, \theta) &= \tfrac{1}{4}\bar{E}[1+\cos(2\theta)]n'(0)+\tfrac{1}{4}[3-\bar{D}-(1+\bar{D})\cos(2\theta)]n''(0)+\sin(2\theta)t''(0)\\
&\quad+\frac{1-\alpha}{4\alpha}\left\{-\sin(2\theta)-2\theta+\pi-\frac{\pi}{2}\bar{D}[1+\cos(2\theta)]\right\}[t'(0)-t''(0)]+o(1),\\
\tau''_{r\theta}(r, \theta) &= \tfrac{1}{4}\bar{E}\sin(2\theta)n'(0)-\tfrac{1}{4}(1+\bar{D})\sin(2\theta)n''(0)-\cos(2\theta)t''(0)\\
&\quad+\frac{1-\alpha}{4\alpha}\left\{\cos(2\theta)+1-\frac{\pi}{2}\bar{D}\sin(2\theta)\right\}[t'(0)-t''(0)]+o(1),
\end{aligned} \tag{4.12}$$

for $0 < \theta < \pi/2$.

Notice that the dominant terms in (4.11) and (4.12) are independent of r. It can be verified that they satisfy all appropriate field equations on D', D'', the continuity conditions at $\theta = 0$, and produce uniform tractions along $\theta = \pm\pi/2$. Therefore, in terms of Cartesian components, the solution corresponding to the *uniform* applied tractions

$$\begin{aligned}
\tau'_{11} &= N', \qquad \tau'_{12} = -T', \qquad (\theta = -\pi/2)\\
\tau''_{11} &= N'', \qquad \tau''_{12} = -T'' \qquad (\theta = \pi/2)
\end{aligned} \tag{4.13}$$

is

$$\begin{aligned}\tau'_{11}(\theta) &= N' - \frac{1+\alpha}{4\alpha}[\sin(2\theta)+2\theta+\pi](T'-T''),\\ \tau'_{22}(\theta) &= -\frac{1}{2}\frac{1-\alpha}{\alpha-2\beta}N' + \frac{1}{2}\frac{1+\alpha}{\alpha-2\beta}N'' + \frac{1+\alpha}{4\alpha}\left[\sin(2\theta)-2\theta+\pi\frac{1-\alpha}{\alpha-2\beta}\right](T'-T''),\\ \tau'_{12}(\theta) &= \frac{1+\alpha}{4\alpha}[1+\cos(2\theta)](T'-T'')-T',\end{aligned} \tag{4.14}$$

for $-\pi/2 < \theta < 0$, and

$$\begin{aligned}\tau''_{11}(\theta) &= N'' - \frac{1-\alpha}{4\alpha}[\sin(2\theta)+2\theta-\pi](T'-T''),\\ \tau''_{22}(\theta) &= -\frac{1}{2}\frac{1-\alpha}{\alpha-2\beta}N' + \frac{1}{2}\frac{1+\alpha}{\alpha-2\beta}N'' + \frac{1-\alpha}{4\alpha}\left[\sin(2\theta)-2\theta+\pi\frac{1+\alpha}{\alpha-2\beta}\right](T'-T''),\\ \tau''_{12}(\theta) &= \frac{1-\alpha}{4\alpha}[1+\cos(2\theta)](T'-T'')-T'',\end{aligned} \tag{4.15}$$

for $0 < \theta < \pi/2$.

If the applied uniform shears are equal, ($T' = T''$), (4.14), (4.15) are independent of θ and we recover the solution for uniform but specifically different biaxial tensions on bonded rectangular regions used by Dundurs [2] to give physical significance to the lines $\alpha = 0$, $\alpha = 2\beta$.

When the applied shears are different ($T' \neq T''$), the shear stress must still be continuous on the bonded edge $\theta = 0$ and (4.14) and (4.15) yield

$$\tau'_{12}(0) = \tau''_{12}(0) = \frac{1-\alpha}{2\alpha}T' - \frac{1+\alpha}{2\alpha}T'', \tag{4.16}$$

which is different from either of the values T', T'' on $\theta = -\pi/2, \pi/2$. Thus the θ-dependent parts in each of (4.14) and (4.15) correspond to asymmetric shear loading solutions on quarter-planes as obtained by Reissner [6].†

The derivation of (4.14) and (4.15) was subject to the restriction $\alpha(\alpha-2\beta) < 0$. These functions clearly are not defined when $\alpha = 0$ or $\alpha = 2\beta$ and this coincides with our findings in case II that the stress field has a logarithmic singularity in those instances. If, however, $T' = T''$ the restriction $\alpha \neq 0$ is not indicated by (4.14) and (4.15) and this is in agreement with (4.6) and (4.7), which show that a logarithmic singularity is not associated with applied shearing tractions that are continuous at $r = 0$.

There appears to be no reason to rule out the validity of the solution for uniform applied tractions given in (4.14) and (4.15) when $\alpha(\alpha-2\beta) > 0$ even though this solution was obtained from the asymptotic considerations only in the case $\alpha(\alpha-2\beta) < 0$. Our failure to arrive at this result when $\alpha(\alpha - 2\beta) > 0$ is related to the assumptions on the behavior of the stress field as $r \to \infty$, which the limiting solution (4.14) and (4.15) does not meet, together with the fact that a zero of $\mathscr{D}(\alpha, \beta; s)$ occurs in the strip $-2 < \mathrm{Re}(s) < -1$ when $\alpha(\alpha - 2\beta) > 0$.

† See also Bogy and Sternberg [5].

We have found that the stress field is of order $r^{-\lambda}$, $\log r$, or 1 as $r \to 0$ according as $\alpha(\alpha - 2\beta) \gtreqless 0$. An example of a practical design question is as follows. Suppose the conditions of the present analysis are satisfied for a composite in which μ', μ'', ν'' are fixed, but for which it is possible to vary ν' by some process. For what values of ν' will the stress field be bounded? If $\alpha < 0$, ($km'' < m'$), we need $\alpha > 2\beta$, or in view of (2.5), $m' > 4(1-k)+km''$. Therefore, and by (2.4), the condition for a bounded stress field, when $\alpha < 0$, becomes

$$\nu' < \begin{cases} k\nu'' & \text{for plane strain} \\ \dfrac{k\nu''}{1+\nu''(1-k)} & \text{for generalized plane stress.} \end{cases} \tag{4.17}$$

For example, if $k = 0{\cdot}8$, $\nu'' = 0{\cdot}3$, this condition is $\nu' < \begin{cases} 0{\cdot}240 \\ 0{\cdot}226 \end{cases}$. When k is significantly different from 1, the region for bounded stress is a relatively small portion of the k-polygon—below the dashed line in Fig. 4.

5. FRACTION OF LOAD TRANSMITTED BETWEEN DISSIMILAR QUARTER-PLANES

Suppose tractions are applied to the boundary of one of the bonded quarter-planes while the boundary of the other remains free of tractions. What fraction of the total applied load is "transmitted to the foundation at infinity" by each quarter-plane? Alternatively, what fraction of the load applied to one quarter-plane is transferred to the other? Here we answer this question for both loading cases—applied normal loads and applied shear loads.

Let $\mathbf{F}_A$ denote the resultant of the tractions applied to the lower quarter-plane D' and $\mathbf{F}_T$ the resultant force transferred from D' to D'' in the same direction as $\mathbf{F}_A$. Then the magnitudes F_A, F_T are given by

$$F_A = \begin{cases} \displaystyle\int_0^\infty n'(r)\,\mathrm{d}r \\ \displaystyle\int_0^\infty t'(r)\,\mathrm{d}r \end{cases}, \qquad F_T = \begin{cases} \displaystyle\int_0^\infty \tau'_{r\theta}(r,0)\,\mathrm{d}r \\ \displaystyle\int_0^\infty \tau'_{\theta\theta}(r,0)\,\mathrm{d}r \end{cases}, \tag{5.1}$$

when the loading applied to D' is a $\begin{Bmatrix} \text{normal} \\ \text{shearing} \end{Bmatrix}$ one. According to (3.6) F_A is just the transform of the load function evaluated at $s = -1$,

$$F_A = \begin{cases} \hat{n}'(-1) \\ \hat{t}'(-1) \end{cases}. \tag{5.2}$$

In order to compute F_T we use (2.1), which with (5.1) gives, after a formal interchange of the order of integration followed by the subsequent integration with respect to r,

$$F_T = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} \hat{\tau}'_{r\theta}(s,0)\left[\frac{r^{-(s+1)}}{-(s+1)}\right]_0^\infty \mathrm{d}s \\ \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} \hat{\tau}'_{\theta\theta}(s,0)\left[\frac{r^{-(s+1)}}{-(s+1)}\right]_0^\infty \mathrm{d}s \end{cases}. \tag{5.3}$$

Thus we obtain F_T from the asymptotic behavior of the integrals in (5.3) as $r \to 0$ and as $r \to \infty$.

As in Section 4 the asymptotic behavior as $r \to 0$ is determined by the singularity of the integrand nearest to and on the left of the path of integration. It readily follows from the methods and results in Section 4 that no contribution to F_T arises from the limit $r \to 0$. The asymptotic behavior as $r \to \infty$ is determined by the residue at the singularity of the integrand in (5.3) nearest to and on the right of the path of integration. Using (3.14) and (3.15) we find that $s = -1$ is a simple pole of the integrands in (5.3). By use of the residue theorem for clockwise integration around the associated closed contour that encloses the simple pole at $s = -1$ we obtain from (5.3)

$$F_T = \lim_{s \to -1} \begin{cases} \hat{\tau}'_{r\theta}(s, 0) \\ \hat{\tau}'_{\theta\theta}(s, 0) \end{cases}. \tag{5.4}$$

In view of (3.14) this may be written as

$$F_T = \lim_{s \to -1} \begin{cases} H_1(\alpha, \beta; s)L_1(s)/2\mathscr{D}(\alpha, \beta; s) \\ G_2(\alpha, \beta; s)L_2(s)/2\mathscr{D}(\alpha, \beta; s) \end{cases}. \tag{5.5}$$

Finally, using (3.2), (3.5), (3.15) and (5.2) in (5.5) we obtain for both loading cases

$$\frac{F_T}{F_A} = \frac{1}{2} - \frac{\alpha[(\pi/2)^2 - 1]}{2[(\pi/2)^2 - \alpha^2]}. \tag{5.6}$$

This is the fraction of load transmitted from D' to D''.

Some remarks about (5.6) are in order. Notice that the fraction of load transmitted depends only on the one composite parameter α. The fraction is the same for applied normal loads as for applied shear loads and therefore by superposition is applicable to any combination of applied normal and shear loads. The fraction does not depend in any way on the particular distribution of the applied tractions. The range of α is $-1 < \alpha < 1$ and (5.6) implies the particular values

$$\frac{F_T}{F_A} \begin{cases} \to 1 & \text{as} \quad \alpha \to -1 & (\text{as} \quad \mu'/\mu'' \to 0) \\ = \frac{1}{2} & \text{when} \quad \alpha = 0 & (\text{when} \quad \mu'/\mu'' = m'/m''). \\ \to 0 & \text{as} \quad \alpha \to 1 & (\text{as} \quad \mu'/\mu'' \to \infty) \end{cases}$$

Therefore the fraction of load transmitted depends *primarily* on the ratio of the two shear moduli, with the more rigid quarter-plane transmitting a larger fraction of the load to the foundation at infinity. This is not precisely the situation, however, since (as can easily be seen in Fig. 3) it is possible to have $\alpha > 0$ for $k < 1$, for instance, in which case the material with the smaller shear modulus assumes a larger fraction of the load.

The computations carried out in this section illustrate the use of the complex integral representation (2.1) to obtain a particular global result. This result would be much more difficult to derive from the corresponding real integral representation.

6. THE CONCENTRATED NORMAL LOAD

In this section the real integral representations for the stress components are derived from the path integrals in (2.1), specific load functions are chosen, and the stress components are computed numerically along the bonded edge for various values of the composite parameters α, β.

Since the load functions $n'(r)$, $t'(r)$, etc., are assumed to be absolutely integrable it follows from (3.6) that their transforms $\hat{n}'(s)$, $\hat{t}'(s)$, etc., are not singular at $s = -1$. It can be verified from (3.11), (3.12) that $s = -1$ is a simple pole of $\hat{\tau}'_{rr}$, $\hat{\tau}''_{rr}$ and is a removable singularity of $\hat{\tau}'_{\theta\theta}$, $\hat{\tau}'_{r\theta}$, $\hat{\tau}''_{\theta\theta}$, $\hat{\tau}''_{r\theta}$. We may use the path of integration in (2.1) determined by $c = -1$ for computing the stress components provided we deform it to the left of $s = -1$ and include the appropriate part of the residue when $s = -1$ is a pole. It is convenient to make a change of complex variable from s to p according to

$$p = -(s+1) = \varphi + i\eta. \tag{6.1}$$

Then the path of integration $\text{Re}(s) = -1$ in the s-plane corresponds to the imaginary axis $\varphi = 0$ in the p-plane (See Fig. 6), and the integrals in (2.1) assume the form

$$\tau_{rr}(r,\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\tau}_{rr}(-1-i\eta,\theta)r^{-1+i\eta}\,d\eta - \frac{1}{2}\operatorname*{Res}_{s=-1}\{\hat{\tau}_{rr}(s,\theta)r^{-(s+2)}\}, \tag{6.2}$$

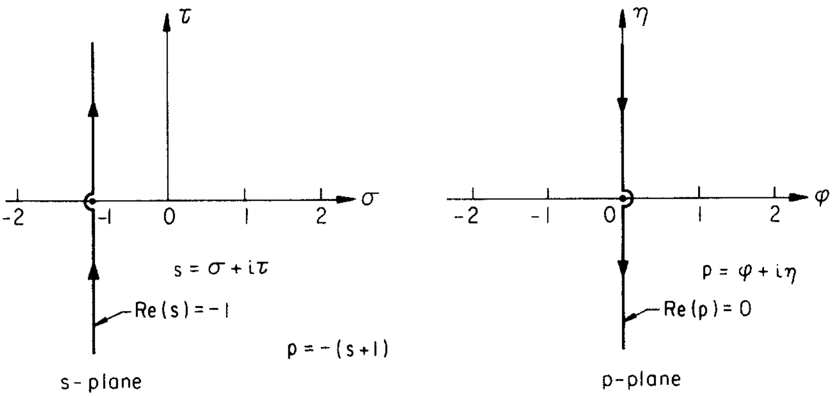

Fig. 6. Path of integration for stress components.

or

$$\begin{aligned}\tau_{rr}(r,\theta) = \frac{1}{2\pi r}\int_{-\infty}^{\infty} &\{[\hat{\tau}^R_{rr}(-1-i\eta,\theta)\cos(\eta\log r) - \hat{\tau}^I_{rr}(-1-i\eta,\theta)\sin(\eta\log r)] \\ &+ i[\hat{\tau}^R_{rr}(-1-i\eta,\theta)\sin(\eta\log r) + \hat{\tau}^I_{rr}(-1-i\eta,\theta)\cos(\eta\log r)]\}\,d\eta \\ &-\tfrac{1}{2}\operatorname*{Res}_{s=-1}\{\hat{\tau}_{rr}(s,\theta)r^{-(s+2)}\},\end{aligned} \tag{6.3}$$

where $\hat{\tau}^R_{rr}$, $\hat{\tau}^I_{rr}$ stand for the real and imaginary parts of $\hat{\tau}_{rr}$. It follows from (3.11) and (3.12) that the first bracket in the integrand of (6.3) contains only even functions of η whereas those in the second bracket are odd, so that (6.3) becomes

$$\begin{aligned}\tau_{rr}(r,\theta) = \frac{1}{\pi r}\int_{0}^{\infty} &[\hat{\tau}^R_{rr}(-1-i\eta,\theta)\cos(\eta\log r) - \hat{\tau}^I_{rr}(-1-i\eta,\theta)\sin(\eta\log r)]\,d\eta \\ &-\tfrac{1}{2}\operatorname*{Res}_{s=-1}\{\hat{\tau}_{rr}(s,\theta)r^{-(s+2)}\}.\end{aligned} \tag{6.4}$$

Similar expressions hold for $\tau_{\theta\theta}$, $\tau_{r\theta}$ in which the residue term vanishes.

The specific loading to be considered here is that of a concentrated normal force of magnitude P applied at $r = d$ on the boundary of D' as depicted in Fig. 7. Thus we assume for the load functions

$$n'(r) = P\delta(r-d), \qquad t'(r) = n''(r) = t''(r) = 0, \tag{6.5}$$

where $\delta(r-d)$ stands for the delta function and is to be defined in the usual manner by the limit of a sequence of well defined load functions with unit resultant. From (3.2) and (6.5) there follows

$$\hat{n}'(s) = Pd^{s+1}, \hat{t}'(s) = \hat{n}''(s) = \hat{t}''(s) = 0. \tag{6.6}$$

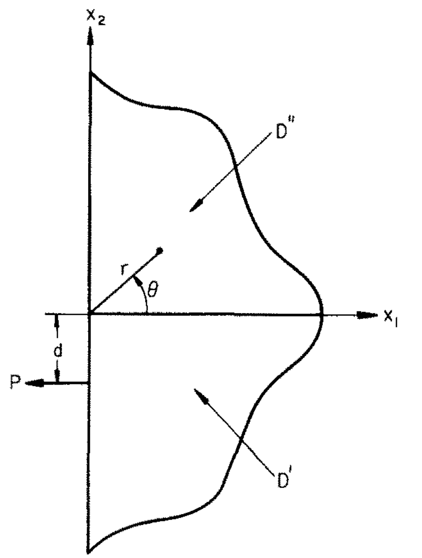

FIG. 7. Bonded quarter-planes under a concentrated normal boundary load.

Next define dimensionless coordinates and stress components by

$$\xi_i = x_i/d, \qquad \rho = r/d, \qquad \sigma_{ij}(\rho, \theta) = \frac{d}{P}\tau_{ij}(r, \theta) \tag{6.7}$$

so that

$$\hat{\sigma}_{ij}(s, \theta) = \int_0^\infty \sigma_{ij}(\rho, \theta)\rho^{s+1}\,\mathrm{d}\rho = \frac{d^{-(s+1)}}{P}\hat{\tau}_{ij}(s, \theta), \tag{6.8}$$

and

$$\hat{\sigma}'_{\theta\theta}(s, -\pi/2) = \frac{d^{-(s+1)}}{P}\hat{n}'(s) = 1. \tag{6.9}$$

Then, in terms of the dimensionless coordinates and stress defined by (6.7), we are considering the simplest problem—one such that the transform of the load function is unity. Corresponding to (6.4) there follows, in the same manner, expressions for $\sigma'_{rr}(\rho, \theta)$, $\sigma''_{rr}(\rho, \theta)$, etc., of the form

$$\sigma_{rr}(\rho,\theta) = \frac{1}{\pi\rho}\int_0^\infty \left[\hat{\sigma}^R_{rr}(-1-i\eta,\theta)\cos(\eta\log\rho) - \hat{\sigma}^I_{rr}(-1-i\eta,\theta)\sin(\eta\log\rho)\right] \mathrm{d}\eta$$
$$-\tfrac{1}{2}\operatorname*{Res}_{s=-1}\{\hat{\sigma}_{rr}(s,\theta)\rho^{-(s+2)}\}. \tag{6.10}$$

We are now in a position to compute the dimensionless stress components along the bonded edge $\theta = 0$ as a function of ρ. In view of (3.5), (3.14) and (6.6) we obtain from (6.8)

$$\begin{aligned} \hat{\sigma}'_{rr}(\alpha,\beta;s,0^-) &= F_1(\alpha,\beta;s)/2s(s+1)\mathscr{D}(\alpha,\beta;s),\\ \hat{\sigma}''_{rr}(\alpha,\beta;s,0^+) &= \bar{F}_3(\alpha,\beta;s)/2s(s+1)\mathscr{D}(\alpha,\beta;s),\\ \hat{\sigma}'_{\theta\theta}(\alpha,\beta;s,0^-) = \hat{\sigma}''_{\theta\theta}(\alpha,\beta;s,0^+) &= G_1(\alpha,\beta;s)/2s(s+1)\mathscr{D}(\alpha,\beta;s),\\ \hat{\sigma}'_{r\theta}(\alpha,\beta;s,0^-) = \hat{\sigma}''_{r\theta}(\alpha,\beta;s,0^+) &= H_1(\alpha,\beta;s)/2s(s+1)\mathscr{D}(\alpha,\beta;s). \end{aligned} \tag{6.11}$$

Letting $I_k(\rho)$ $(k = 1, 2, 3, 4)$ denote the functions

$$\begin{aligned} I_1(\rho) &= \sigma'_{rr}(\rho,0^-), \qquad I_2(\rho) = \sigma''_{rr}(\rho,0^+),\\ I_3(\rho) &= \sigma'_{\theta\theta}(\rho,0^-) = \sigma''_{\theta\theta}(\rho,0^+),\\ I_4(\rho) &= \sigma'_{r\theta}(\rho,0^-) = \sigma''_{r\theta}(\rho,0^+), \end{aligned} \tag{6.12}$$

there follows from (3.2), (3.15), (6.10) and (6.11)

$$I_k(\rho) = \frac{1}{2\pi\rho}\int_0^\infty [A_k(\eta)\cos(\eta\log\rho) + B_k(\eta)\sin(\eta\log\rho)]\,\mathrm{d}\eta + R_k(\rho), \tag{6.13}$$

in which

$$\begin{aligned} A_1(\eta) &= [(3\alpha-4\beta+1)(\alpha-\beta)\eta^3 + (1+\alpha)^2\eta\\ &\quad + (3\alpha-4\beta+1)(1+\beta)\eta\sinh^2(\eta\pi/2)]\sinh(\eta\pi/2)/D(\eta),\\ B_1(\eta) &= -\{[(\alpha-4\beta+3)(\alpha-\beta) - \alpha(1-\alpha)]\eta^2 + 2(1+\alpha)\\ &\quad + [(\alpha-4\beta+3)(1+\beta) - (1-\alpha)]\sinh^2(\eta\pi/2)\}\sinh(\eta\pi/2)/D(\eta),\\ A_2(\eta) &= (1-\alpha)[(\alpha-\beta)\eta^3 - (\alpha-4\beta-1)\eta + (1+\beta)\eta\sinh^2(\eta\pi/2)]\sinh(\eta\pi/2)/D(\eta),\\ B_2(\eta) &= -(1-\alpha)[(2\alpha-3\beta)\eta^2 + 2 + (2-\beta)\sinh^2(\eta\pi/2)]\sinh(\eta\pi/2)/D(\eta),\\ A_3(\eta) &= -(1-\alpha)[(\alpha-\beta)\eta^3 + (1+\alpha)\eta + (1+\beta)\eta\sinh^2(\eta\pi/2)]\sinh(\eta\pi/2)/D(\eta),\\ B_3(\eta) &= -(1-\alpha)\beta[\eta^2 - \sinh^2(\eta\pi/2)]\sinh(\eta\pi/2)/D(\eta),\\ A_4(\eta) &= -(1-\alpha)[\alpha\eta^2 + \sinh^2(\eta\pi/2)]\cosh(\eta\pi/2)/D(\eta),\\ B_4(\eta) &= -(1-\alpha)[(\alpha-\beta)\eta^3 + (1+\beta)\eta\sinh^2(\eta\pi/2)]\cosh(\eta\pi/2)/D(\eta), \end{aligned} \tag{6.14}$$

where

$$D(\eta) = (\beta^2-1)\sinh^4(\eta\pi/2) + [2\beta(\alpha-\beta)\eta^2 - 1]\sinh^2(\eta\pi/2) + (\alpha-\beta)^2\eta^4 + \alpha^2\eta^2, \tag{6.15}$$

and

$$R_k(\rho) = \frac{\pi}{4\rho}\left[\frac{1-(-1)^k\alpha}{(\pi/2)^2-\alpha^2}\right], \qquad k = 1, 2, \qquad R_3(\rho) = R_4(\rho) = 0. \tag{6.16}$$

The improper integrals in (6.13) exist for fixed $0 < \rho < \infty$ and are readily computed by appropriate quadrature methods. Since $A_k(\eta)$, $B_k(\eta)$ are of order $\eta\exp(-\eta\pi/2)$ or less as $\eta \to \infty$ the integrals can be closely approximated with a reasonably small finite upper limit N. $A_k(\eta)$ are of order 1 as $\eta \to 0$ and $B_3(\eta), B_4(\eta) \to 0$ as $n \to 0$ but

$$B_k(\eta) = \left[\frac{1-(-1)^k\alpha}{(\pi/2)^2-\alpha^2}\right]\frac{\pi}{\eta}+O(\eta) \quad \text{as} \quad \eta \to 0, \qquad k = 1, 2. \tag{6.17}$$

Using the well-known integral,

$$\int_0^\infty \frac{\sin(\eta \log \rho)}{\eta}\,\mathrm{d}\eta = \pm\pi/2, \qquad \log\rho \gtrless 0, \tag{6.18}$$

with (6.16) we can replace $I_1(\rho)$, $I_2(\rho)$ in (6.13) by

$$\begin{aligned} I_k(\rho) = {} & \frac{1}{2\pi\rho}\int_0^\infty [A_k(\rho)\cos(\eta\log\rho)+B_k'(\eta)\sin(\eta\log\rho)]\,\mathrm{d}\eta \\ & +\frac{\pi}{4\rho}\left[\frac{1-(-1)^k\alpha}{(\pi/2)^2-\alpha^2}\right](1\pm 1), \qquad \rho \gtrless 1, \qquad k = 1, 2, \end{aligned} \tag{6.19}$$

where

$$B_k'(\eta) = B_k(\eta)-\left[\frac{1-(-1)^k\alpha}{(\pi/2)^2-\alpha^2}\right]\frac{\pi}{\eta}. \tag{6.20}$$

In view of (6.17) and (6.20) $B_k'(\eta) \to 0$ as $\eta \to 0$, which improves the convergence of the integral in $I_k(\rho)$, $k = 1, 2$, as far as the lower limit is concerned, but clearly makes matters worse at the upper limit since $B_k'(\eta)$ is of order η^{-1} as $\eta \to \infty$ instead of order $\exp(-\eta\pi/2)$. This situation can be remedied as explained in [7] in the following manner:

$$\int_0^\infty B_k'(\eta)\sin(\eta\log\rho)\,\mathrm{d}\eta = \int_0^N B_k'(\eta)\sin(\eta\log\rho)\,\mathrm{d}\eta + \int_N^\infty B_k'(\eta)\sin(\eta\log\rho)\,\mathrm{d}\eta \tag{6.21}$$

and by (6.20)

$$\int_N^\infty B_k'(\eta)\sin(\eta\log\rho)\,\mathrm{d}\eta = \int_N^\infty B_k(\eta)\sin(\eta\log\rho)\,\mathrm{d}\eta - \pi\left[\frac{1-(-1)^k\alpha}{(\pi/2)^2-\alpha^2}\right]\int_N^\infty \frac{\sin(\eta\log\rho)}{\eta}\,\mathrm{d}\eta. \tag{6.22}$$

In terms of the sine integral function Si( )

$$\int_N^\infty \frac{\sin(\eta\log\rho)}{\eta}\,\mathrm{d}\eta = \pm\frac{\pi}{2}-\mathrm{Si}(N\log\rho), \qquad \rho \gtrless 1. \tag{6.23}$$

Using (6.21)–(6.23) we obtain from (6.19)

$$I_k(\rho) = \frac{1}{2\pi\rho}\int_0^N [A_k(\eta)\cos(\eta\log\rho)+B_k'(\eta)\sin(\eta\log\rho)]\,d\eta$$
$$+\frac{\pi}{4\rho}\left\{\frac{1-(-1)^k\alpha}{(\pi/2)^2-\alpha^2}\right\}\left[1+\frac{2}{\pi}\,\mathrm{Si}(N\log\rho)\right] \tag{6.24}$$
$$+\frac{1}{2\pi\rho}\int_N^\infty [A_k(\eta)\cos(\eta\log\rho)+B_k(\eta)\sin(\eta\log\rho)]\,d\eta, \qquad k = 1, 2.$$

As indicated previously the last term can be made arbitrarily small for fixed ρ by choosing N large enough. Acceptable accuracy was achieved for all values of ρ by neglecting the last term in (6.24) with $N = 6$.

It is of interest to note that for fixed N the integrals in (6.24) approach zero as $\rho \to \infty$ while $\mathrm{Si}(N\log\rho) \to \pi/2$ so that

$$I_k(\rho) \to \frac{\pi}{2\rho}\left[\frac{1-(-1)^k\alpha}{(\pi/2)^2-\alpha^2}\right]$$

as $\rho \to \infty, k = 1, 2$, which is twice $R_k(\rho)$ in (6.16) and hence by (6.10) agrees with the asymptotic values of $I_k(\rho)$ obtained by residue considerations.

The asymptotic behavior of $I_k(\rho)$ as $\rho \to 0$ is easily computed from the results in Section 4. Since it was found there that these functions can be of order $\rho^{-\lambda}$ as $\rho \to 0$ for arbitrarily small $\lambda > 0$ they may differ considerably from the first term of their asymptotic expansions for unusually small values of ρ. Therefore it is necessary to compute $I_k(\rho)$ from (6.13) and (6.24) for very small ρ ($\rho = 0{\cdot}01$ say). The argument of the trigonometric functions in the integrals can then be quite large in magnitude and the resulting rapid oscillations makes it necessary to use a quadrature method for the numerical integration that takes this into consideration. The method used here is the one attributed to Filon and explained in detail by Tranter [7].

A useful guide to the accuracy of the numerical computations is provided by the elementary expressions for $I_k(\rho)$ for the case when the two materials are identical ($\alpha = \beta = 0$). In this instance $A_k(\eta)$, $B_k(\eta)$ in (6.14) are given by

$$A_1(\eta) = A_2(\eta) = -A_3(\eta) = -\eta B_1(\eta)/2 = -\eta B_2(\eta)/2 = -\eta/\sinh(\eta\pi/2),$$
$$B_3(\eta) = 0, \qquad A_4(\eta) = B_4(\eta)/\eta = 1/\cosh(\eta\pi/2). \tag{6.25}$$

Using the elementary integrals

$$\int_0^\infty \frac{\eta\cos(\eta x)\,d\eta}{\sinh(\eta\pi/2)} = \mathrm{sech}^2(x), \qquad \int_0^\infty \frac{\sin(\eta x)\,d\eta}{\sinh(\eta\pi/2)} = \tanh(x),$$
$$\int_0^\infty \frac{\cos(\eta x)\,d\eta}{\cosh(\eta\pi/2)} = \mathrm{sech}(x), \qquad \int_0^\infty \frac{\eta\sin(\eta x)\,d\eta}{\cosh(\eta\pi/2)} = \sinh(x)\,\mathrm{sech}^2(x), \tag{6.26}$$

and (6.25) in (6.13) we recover the expressions

$$\sigma_{rr}(\rho, 0) = \rho^2\sigma_{\theta\theta}(\rho, 0) = \rho\sigma_{r\theta}(\rho, 0) = \frac{2}{\pi}\frac{\rho^3}{(1+\rho^2)^2}, \tag{6.27}$$

which are the stress components along the axis due to a unit concentrated normal load applied to a half-plane a unit distance below the axis. These functions serve also as a base for comparison when α, β take values other than zero.

The result of the computations of the $I_k(\rho)$ defined by (6.12) using the techniques explained above are exhibited in Figs. 8–11 for various α, β. The α, β chosen for computation represent typical values in each of the three regions $\alpha(\alpha-2\beta) \gtreqless 0$, which characterize the different classes of singular stress fields discussed in Section 4. The chosen (α, β) values satisfying $\alpha(\alpha-2\beta) > 0$ were $(-0{\cdot}5, 0)$, $(0{\cdot}5, 0)$, those satisfying $\alpha(\alpha-2\beta) = 0$ were $(0, 0)$, $(0, 0{\cdot}1)$, $(-0{\cdot}5, -0{\cdot}25)$, and those satisfying $\alpha(\alpha-2\beta) < 0$ were $(0{\cdot}1, 0{\cdot}1)$. In Fig. 5 these points are represented by heavy dots.

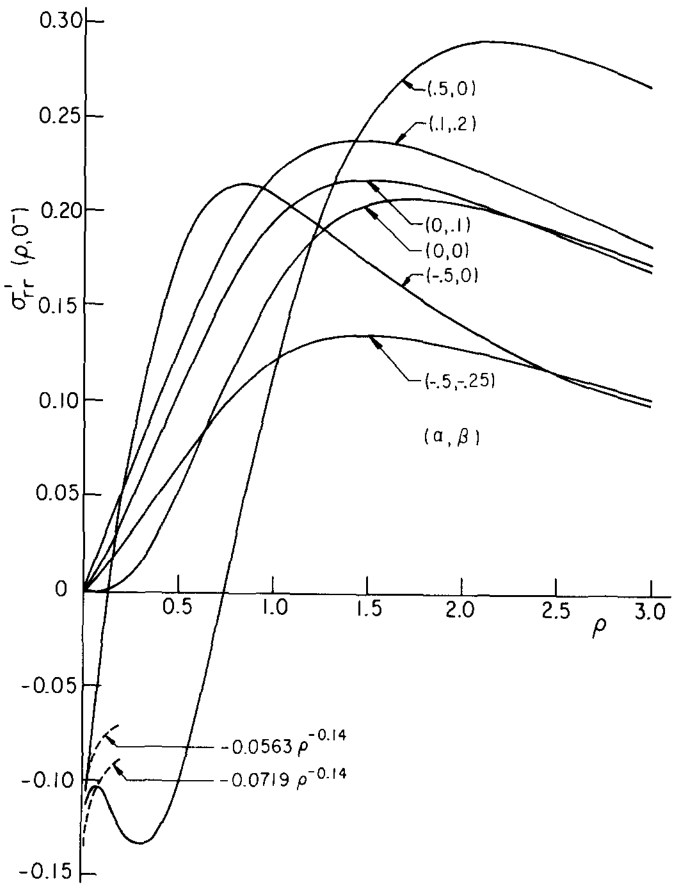

FIG. 8. Dependence of σ'_{rr} on ρ at $\theta = 0^-$ for various α, β.

Figs. 8 and 9 show $\sigma'_{rr}(\rho, 0^-)$, $\sigma''_{rr}(\rho, 0^+)$, respectively, for each of the above mentioned values of α, β. In view of the particular loading under consideration all of $n'(0)$, $t'(0)$, $n''(0)$, $t''(0)$ vanish and hence by (4.6), (4.7), (4.11) and (4.12) the curves in Figs. 8 and 9 approach the origin as $\rho \to 0$ for α, β satisfying $\alpha(\alpha-2\beta) \lesssim 0$. For the cases $(0{\cdot}5, 0)$, $(-0{\cdot}5, 0)$, satisfying $\alpha(\alpha-2\beta) > 0$, we read the value $\lambda \cong 0{\cdot}14$ from Fig. 5. Thus, and by (4.1), (4.2) with (3.14),

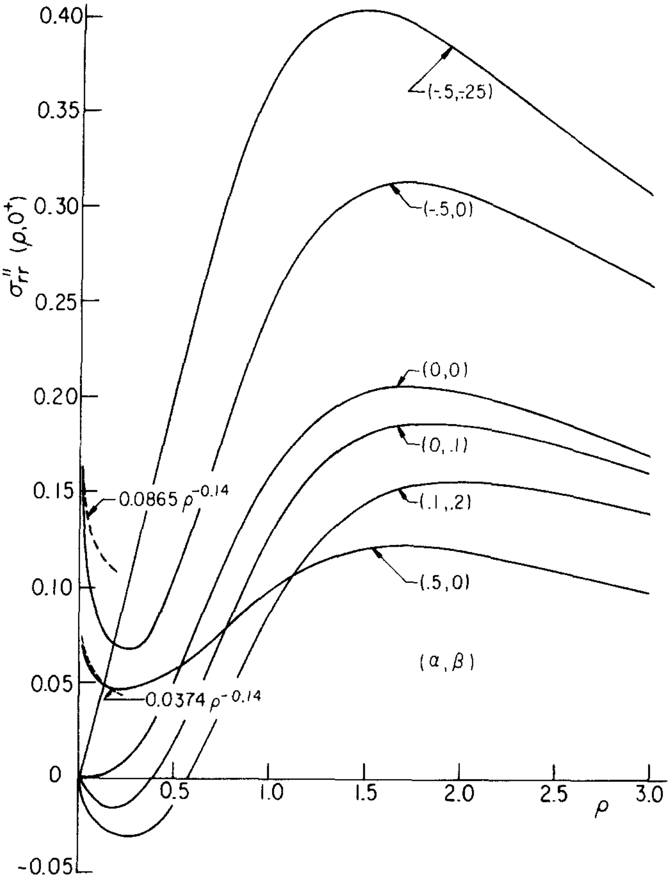

FIG. 9. Dependence of σ''_{rr} on ρ at $\theta = 0^+$ for various α, β.

(3.15) and (6.8), (6.9), we obtain as $\rho \to 0$

$$\begin{aligned} \sigma'_{rr}(\rho, 0^-) &= -0{\cdot}0719\rho^{-0{\cdot}14} + o(\rho^{-0{\cdot}14}), \\ \sigma''_{rr}(\rho, 0^+) &= 0{\cdot}0374\rho^{-0{\cdot}14} + o(\rho^{-0{\cdot}14}), \end{aligned} \tag{6.28}$$

for $(\alpha, \beta) = (0{\cdot}5, 0)$ and

$$\begin{aligned} \sigma'_{rr}(\rho, 0^-) &= -0{\cdot}0563\rho^{-0{\cdot}14} + o(\rho^{-0{\cdot}14}), \\ \sigma''_{rr}(\rho, 0^+) &= 0{\cdot}0865\rho^{-0{\cdot}14} + o(\rho^{-0{\cdot}14}), \end{aligned} \tag{6.29}$$

for $(\alpha, \beta) = (-0{\cdot}5, 0)$. The corresponding curves in Figs. 8 and 9 exhibit this asymptotic behavior as $\rho \to 0$ but differ considerably from this one term expansion (indicated by dashed lines) at values of ρ as small as 0·1. For composites satisfying $\alpha(\alpha - 2\beta) \lesseqgtr 0$ the curves are qualitatively similar to that for identical materials, $(\alpha, \beta) = (0, 0)$, as computed from (6.27). But the curves for composites satisfying $\alpha(\alpha - 2\beta) > 0$ differ markedly near $\rho = 0$ from that for identical materials.

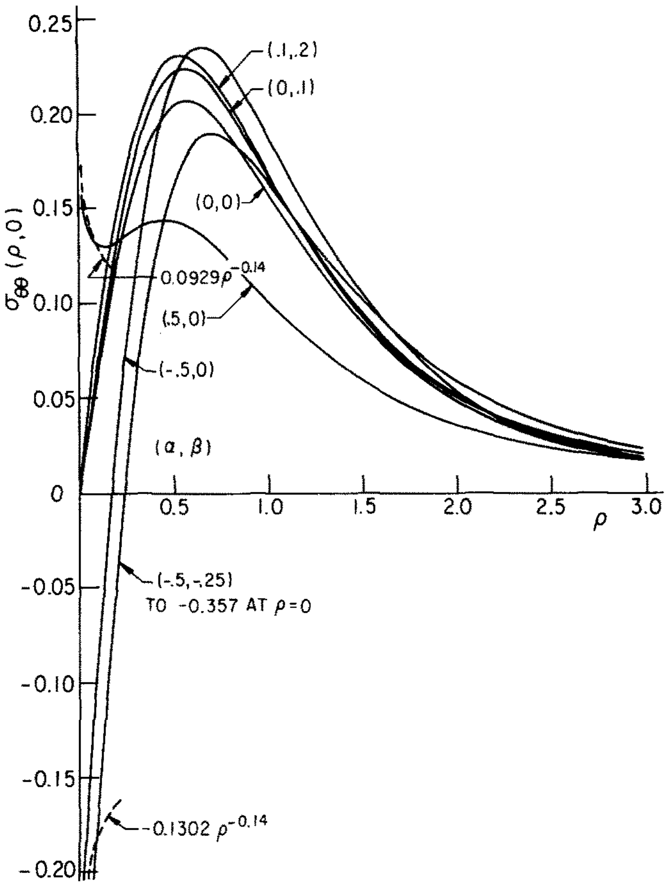

FIG. 10. Dependence of $\sigma_{\theta\theta}$ on ρ at $\theta = 0$ for various α, β.

As stated previously in connection with (6.24)

$$\left.\begin{matrix}\sigma'_{rr}(\rho, 0^-)\\ \sigma''_{rr}(\rho, 0^+)\end{matrix}\right\} = \frac{\pi}{2\rho}\left\{\frac{1 \pm \alpha}{(\pi/2)^2 - \alpha^2}\right\} + o(\rho^{-1}) \quad \text{as} \quad \rho \to \infty. \tag{6.30}$$

The functions approach this asymptote quite rapidly as ρ increases and are well represented by this term for $\rho > 5$.

Figs. 10 and 11 show $\sigma_{\theta\theta}(\rho, 0)$, $\sigma_{r\theta}(\rho, 0)$ respectively and most of the above comments apply to these functions as well. In particular, as $\rho \to 0$

$$\begin{aligned}\sigma_{\theta\theta}(\rho, 0) &= 0{\cdot}0929\rho^{-0\cdot14} + o(\rho^{-0\cdot14}),\\ \sigma_{r\theta}(\rho, 0) &= 0{\cdot}0240\rho^{-0\cdot14} + o(\rho^{-0\cdot14}),\end{aligned} \tag{6.31}$$

for $(\alpha, \beta) = (0{\cdot}5, 0)$ and

$$\begin{aligned}\sigma_{\theta\theta}(\rho, 0) &= -0{\cdot}1302\rho^{-0\cdot14} + o(\rho^{-0\cdot14}),\\ \sigma_{r\theta}(\rho, 0) &= 0{\cdot}0325\rho^{-0\cdot14} + o(\rho^{-0\cdot14}),\end{aligned} \tag{6.32}$$

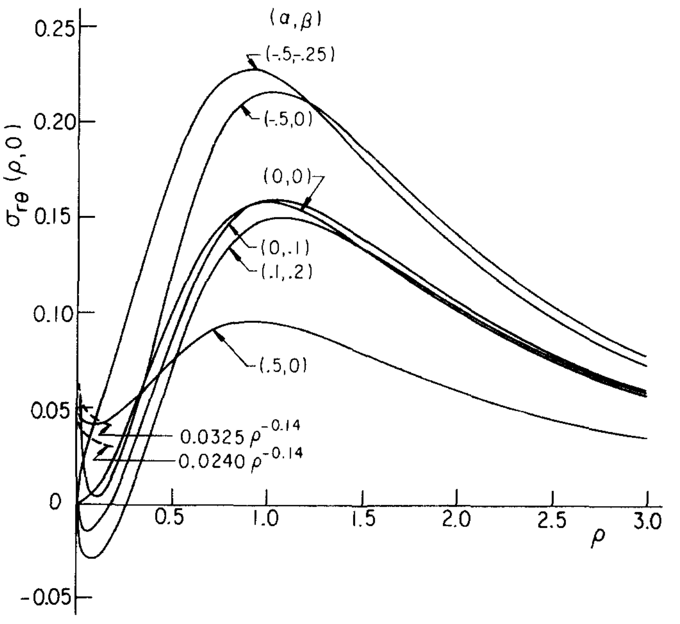

FIG. 11. Dependence of $\sigma_{r\theta}$ on ρ at $\theta = 0$ for various α, β.

for $(\alpha, \beta) = (-0{\cdot}5, 0)$, whereas these curves approach the origin for all of the other values of α, β shown except $(-0{\cdot}5, -0{\cdot}25)$. From (4.4), (6.6) we have in the present example

$$n'(0) = 0, \qquad N'(-2) = P/d. \tag{6.33}$$

Thus, and by (6.7) with (4.5)–(4.7), we obtain when $\alpha(\alpha-2\beta) = 0$

$$\sigma'_{\theta\theta}(0, 0^-) = \sigma''_{\theta\theta}(0, 0^+) = -2A = \frac{4\alpha(1-2\beta)}{16\beta^2+\pi^2(1-\alpha^2)}, \tag{6.34}$$

which has the value $-0{\cdot}357$ when $\alpha, \beta = (-0{\cdot}5, -0{\cdot}25)$ as is indicated in Fig. 10.

According to (5.1) the areas under the curves in Fig. 11 represent the load transmitted from D' to D''. The results shown in Fig. 11 agree qualitatively with the results of Section 5, i.e. roughly speaking, the area increases as $\alpha \to -1$ and decreases as $\alpha \to 1$.

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Абстракт—Дается в конечном виде выражение для поля напряжений в соединенных четвертях плоскостей, изготовленных из разных упругих материалов, под влиянием произвольной /интегрируемой/ нормальной нагрузки и нагрузки сдвига, приложенных на границе. Это выражение представленно с помощью двух результирующих параметров α, β, приведенных в дискусии Дандера [2], автора предыдущей работы [1].

Исследуется поле напряжений для всех физически относительных значений α, β. Порядок сингулярности является $r^{-\lambda}$, $\log r$, или 1, в зависимости от значений α, β. Из предельного случая получается также решение для равномерных нормальных сил сцеплиния и такихже сил сдвига.

Используется, также, общее решение для определения несложной алгебраической функции только для α, которая представляет часть приложенной нагрузки вызванной каждой четвертью плоскости.

Окончательно, нагрузка сводится к концентрической нормальной силе. Остаточные компоненты напряжений даются графически в виде функции положения, вдоль края соединения, для разных значений α, β.